# The Classical Subspace of Quantum State Space:

## Geometry, Canonical Dynamics, and Error Correction

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## Abstract

We identify V_eigenvalue(ρ) — the kernel of the modular Hamiltonian log Δ_ρ of Tomita–Takesaki theory, restricted to finite dimensions — as the canonical classical subspace of the density matrix manifold D. This subspace admits nine independent characterisations (Theorem VEIGEN): as the kernel of the Petz recovery map generator, the locus where all Petz quantum monotone metrics collapse to the classical Fisher–Rao metric, the space of directions that generate zero quantum coherence, and six further geometric and algebraic characterisations. The subspace forms a smooth real-analytic subbundle of constant rank on each eigenvalue-degeneracy stratum of int(D), and these strata assemble into a Whitney stratification (Theorem STRAT). The rank jumps between strata encode quantum-classical phase transitions: each eigenvalue merger transfers one classical degree of freedom from V_eigenvalue to V_degenerate.

On this canonically defined subspace, two fundamental ambiguities of quantum information geometry are simultaneously resolved: (1) all Petz quantum monotone metrics collapse to Fisher–Rao (Theorem PETZ), and (2) the unique canonical generator is (M*) = ρ log ρ − Tr(ρ log ρ)ρ, characterised axiomatically (Theorem 3A), variationally as the maximum Fisher–Rao efficiency flow (Theorem 3B), and via modular equivariance (Theorem COEFF). The (M*) flow converges superexponentially to pure states for all N ≥ 2 (Theorem CONV-ALL: ε(τ) ≤ exp(−exp(cτ)) with c ≥ 1/2 for all N, c = 1 for N = 2) while generating zero quantum coherence at every instant (Theorem ZEROCOHERENCE).

The central consequence for quantum error correction is Theorem V-GENERAL: for any quantum channel N (without symmetry assumptions), V_eigenvalue errors satisfy diag_σ(N(ρ)) = N^cl · λ(ρ), where N^cl is the classical action matrix. This elementary identity implies that V_eigenvalue errors require zero quantum resources — classical post-processing is universally sufficient. The canonical decomposition T_ρD = V_eigenvalue ⊕ V_unitary ⊕ V_degenerate therefore partitions quantum errors into classically correctable (V_eigenvalue: zero quantum advantage) and quantum-correctable (V_unitary ⊕ V_degenerate: quantum operations may help) components, determined by the geometry of state space, not by the specific error model or code.

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## 1. Introduction

A fundamental question in quantum information geometry [2, 3] is whether the density matrix manifold D possesses canonical subspaces with distinguished mathematical and operational properties. This paper answers affirmatively, identifying one such subspace — V_eigenvalue(ρ) — and establishing its complete characterisation.

$V\_eigenvalue(\rho) = \{A \in T\_\rho D : [A, \rho] = 0, \mathrm{Tr}\, A = 0\}$ consists of those tangent directions that alter the eigenvalues of ρ while leaving its eigenvectors unchanged. Its characterisation as $\ker(\log \Delta\_\rho)$ — the kernel of the finite-dimensional modular Hamiltonian — connects it directly to the Tomita–Takesaki modular theory of von Neumann algebras, and establishes it as the natural finite-dimensional incarnation of the 'classical part' of modular flow.

The paper has three main contributions. First, we show that V_eigenvalue is simultaneously characterised by nine independent mathematical structures (Theorem VEIGEN), most importantly as: the kernel of the Petz recovery map generator (Theorem PETZ-FIXED), the locus of zero quantum coherence generation (Theorem ZEROCOHERENCE), and the subspace on which all Petz quantum monotone metrics coincide with classical Fisher–Rao (Theorem PETZ). Second, we characterise the unique canonical generator (M*) of V_eigenvalue dynamics by axioms, variational optimality, and modular equivariance. Third, we prove that V_eigenvalue errors require zero quantum resources for correction in any quantum channel (Theorem V-GENERAL), providing the first geometric characterisation of the classical-quantum boundary in quantum error correction.

Relation to Lindblad's theorem [12]: Lindblad [12] and Gorini–Kossakowski–Sudarshan [9] characterise the maximal class of completely positive trace-preserving (CPTP) generators — the outer boundary of quantum irreversibility (see also Breuer–Petruccione [5] for background). This paper identifies the canonical generator of the geometrically distinguished V_eigenvalue subclass — neither CPTP nor a restriction of any GKSL flow, but the unique canonical element of the 'classical sector' within T_ρD. Together they bound quantum irreversibility from above (Lindblad–GKS) and identify the canonical classical axis within (this paper).

## 1.1 Related Structures and Distinctions

V_eigenvalue is related to, but distinct from, several structures studied in the literature. The following comparison addresses potential confusion before the technical development begins.

| **Structure** | **Definition / Condition** | **Relation to V_eigenvalue** |
|---|---|---|
| Decoherence-free subspace (DFS) | States invariant under a specific noise channel N | Distinct. DFS depends on N and lives in Hilbert space; V_eigenvalue is a tangent-space structure intrinsic to ρ alone, defined without reference to any channel. |
| Noiseless subsystem | Subsystem algebra fixed by a gauge group of noise operators | Distinct. Noiseless subsystems require a specific Kraus representation; V_eigenvalue coincides with a noiseless subsystem only when ρ commutes with all noise operators, a non-generic condition. |
| Classical-diagonal subspace | Matrices diagonal in a fixed basis | Coincides with V_eigenvalue only when the fixed basis equals ρ’s eigenbasis. V_eigenvalue is basis-free: it is defined intrinsically by the modular Hamiltonian of ρ and rotates with |

| Structure | Definition / Condition | Relation to V_eigenvalue |
|---|---|---|
| | | ρ. |
| Commutant of ρ | $\{A : [A,\rho]=0\}$, dimension N in non-degenerate case | V_eigenvalue = commutant ∩ $T_\rho D \cap \{\mathrm{Tr}=0\}$. The trace-zero and tangent-space conditions select the physically meaningful $(N-1)$-dimensional subspace of directions that actually move ρ. |

Regarding the axiom system (C1)–(C5) for the canonical generator M*: each axiom is necessary in the sense that removing any single condition strictly enlarges the solution set. Specifically, without (C5) (the affine form condition), any flow of the form $d\mu_i/d\tau = \Sigma_j a_{ij}(\rho)g_j$ with zero-column-sum matrix a(ρ) satisfies the remaining conditions, giving an infinite-dimensional solution space. Without (C4’) (the purity-increase condition), both (M*) and −(M*) are solutions. Without (C6) (the normalisation), any positive scalar multiple α(M*) solves the system; Conjecture 2 (Open Problem 6) asks whether (C1)(C2)(C3)(C4’) alone determine F = α(M*) up to positive scaling, and remains open. The two-layer architecture of (C5) is made explicit in Theorem C5-ARCH (Section 6.2): Layer 1 (pointwise affine form) follows from variational optimality (Theorem 3B); Layer 2 (coefficient constancy) follows independently from the boundary structure of Σ (Lemma C5). These are logically independent derivations.

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## 2. Setup and Notation

Let H be a finite-dimensional complex Hilbert space, dim H = N < ∞. $D = \{\rho \in L(H) : \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\,\rho = 1\}$ with Hilbert–Schmidt inner product $\langle A,B\rangle = \mathrm{Tr}(A^\dagger B)$. For ρ ∈ int(D): spectral decomposition $\rho = \Sigma_i \lambda_i P_i$ (general, allowing degeneracy). In the non-degenerate case $P_i = |e_i\rangle\langle e_i|$; in the degenerate case $P_\alpha$ has rank $m_\alpha$.

Notation: $g_i = \log \lambda_i + S(\rho)$ where $S(\rho) = -\Sigma_i \lambda_i \log \lambda_i$; $\Sigma_i \lambda_i g_i = 0$. Eigenvalue simplex $\Sigma = \{\lambda : \lambda_i > 0, \Sigma\lambda_i = 1\}$. Modular operator: $\Delta_\rho(A) = \rho A \rho^{-1}$; $\log \Delta_\rho(A) = [\log \rho, A]$. Stabiliser group: $\mathrm{Stab}(\rho) = \{U \in U(N) : U\rho U^\dagger = \rho\} = \Pi_\alpha U(m_\alpha)$. Fisher–Rao metric on Σ: $g^{FR}_{ij} = \delta_{ij}/\lambda_i$. For channel N and reference σ: classical action matrix $N^{cl}_{ij} := \mathrm{Tr}(N(P_j(\sigma))P_i(\sigma))$; σ-measurement distribution $p_i(\rho|\sigma) := \mathrm{Tr}(\rho P_i(\sigma))$.

The simple-spectrum stratum int_s(D) = {ρ ∈ int(D) : all eigenvalues distinct} is open and dense in int(D).

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## 3. Theorem STRAT: Whitney Stratification of int(D) by V_eigenvalue

This section proves that the subbundle non-smoothness of V_eigenvalue at degenerate points — which could appear as a defect — is in fact the most geometrically rich structure in the paper: a Whitney stratification encoding quantum-classical phase transitions.

### 3.1 The Stratification

Define the eigenvalue degeneracy stratification of int(D): for each ordered partition $(m_1,\ldots,m_k)$ of N with $m_1 \geq \ldots \geq m_k \geq 1$,

$$D_{(m_1,\ldots,m_k)} := \{\rho \in \mathrm{int}(D) : \text{eigenvalue multiplicities are exactly } (m_1,\ldots,m_k)\}$$

The top stratum $D_{(1,...,1)} = int_s(D)$ (N ones, simple spectrum) is open and dense. The bottom stratum $D_{(N)} = \{I/N\}$ is a single point (the maximally mixed state). There are finitely many strata — one for each partition of N.

Each stratum $D_{(m_1,...,m_k)}$ is a smooth embedded submanifold of int(D) of codimension $\Sigma\alpha(m\alpha(m\alpha-1)/2)$. It is defined by: (a) the system of polynomial equations asserting that the characteristic polynomial has degeneracy pattern $(m_1,...,m_k)$ — equivalently, by the vanishing of appropriate minors of the resultant matrix; and (b) the non-vanishing of equations asserting no further mergers. These are real-analytic conditions, making each stratum a real-analytic submanifold.

## 3.2 The Theorem

**Theorem STRAT (Stratified Subbundle Structure of V_eigenvalue).**

The assignment $\rho \mapsto$ V_eigenvalue($\rho$) on int(D) satisfies:

(A) [Stratum-wise smoothness] On each $D_{(m_1,...,m_k)}$, the assignment is a smooth real-analytic vector subbundle of $T(D_{(m_1,...,m_k)})$ of constant rank $k-1$.

(B) [Whitney stratification] The collection $\{D_{(m_1,...,m_k)}\}$ forms a Whitney stratification of int(D): both Whitney conditions (a) and (b) hold at all stratum boundaries.

(C) [Canonical flat connection] On each stratum, V_eigenvalue carries a canonical flat connection $\nabla^V$ with trivial holonomy. The leaf space is $\Sigma_{(k)} = \{(\lambda_1,...,\lambda_k) : \lambda\alpha > 0, \Sigma\lambda\alpha = 1\}$.

(D) [Rank-jump formula] At $\rho_0 \in D_{(m'_1,...,m'_{k_0})}$ approached from $D_{(m_1,...,m_k)}$ with $k_0 < k$: rank V_eigenvalue($\rho_0$) = $k_0 - 1$. The rank drop $k - k_0$ equals the number of eigenvalue mergers.

(E) [Limit theorem] For $\rho(t) \to \rho_0$ from $D_{(m_1,...,m_k)}$: lim V_eigenvalue($\rho(t)$) $\supset$ V_eigenvalue($\rho_0$). The excess $k-k_0$ dimensions converge to components of V_degenerate($\rho_0$).

*Proof of (A). Within $D_{(m_1,...,m_k)}$, no eigenvalue crossing occurs, so the k distinct eigenvalue functions $\lambda_1(\rho) > ... > \lambda_k(\rho)$ are given by real-analytic functions of the matrix entries of $\rho$ (by the analytic implicit function theorem applied to the characteristic polynomial). The spectral projectors $P_\alpha(\rho) = \Pi_{\beta\neq\alpha}(\rho - \lambda_\beta(\rho)I)/(\lambda_\alpha(\rho) - \lambda_\beta(\rho))$ are therefore real-analytic. Hence V_eigenvalue($\rho$) = span$\{c_1P_1+...+c_kP_k : \Sigma c\alpha=0\}$ is a smooth real-analytic subbundle of rank $k-1$.* □

*Proof of (B). Whitney condition (a) (tangent space inclusion at boundary): Let $\rho_n \to \rho_0$ with $\rho_n \in D_{(m_1,...,m_k)}$, $\rho_0 \in D_{(m'_1,...,m'_{k_0})}$. Any tangent vector v to $D_{(m'_1,...,m'_{k_0})}$ at $\rho_0$ decomposes as v = v_eigen + v_uni + v_deg under $T_{\rho_0}D$ = V_eigenvalue($\rho_0$) ⊕ V_unitary($\rho_0$) ⊕ V_degenerate($\rho_0$) (Theorem 3'). Each component lies in $T_{\rho_0}(D_{(m'_1,...,m'_{k_0})})$ since $D_{(m'_1,...,m'_{k_0})}$ is a smooth manifold. Hence lim $T_{\rho_n}(D_{(m_1,...,m_k)}) \supset T_{\rho_0}(D_{(m'_1,...,m'_{k_0})})$.* □

Whitney condition (b) (secant-tangent condition): The strata are defined by real-analytic equations (vanishing patterns of eigenvalue discriminants). By Lojasiewicz [13] — which establishes that real-analytic stratifications of semi-analytic sets satisfy Whitney (b) regularity — the stratification $\{D_{(m_1,...,m_k)}\}$ satisfies Whitney (b). Explicitly: the eigenvalue discriminant $\Delta(\rho) = \Pi_{i<j}(\lambda_i(\rho) - \lambda_j(\rho))^2$ is a real-analytic function of the matrix entries, and its zero locus defines the stratum boundaries as a semi-analytic set. Whitney (b) for such sets follows from [13]; see also Whitney [20] for the foundational theory of stratified spaces. □

*Proof of (C). Define parallel transport along a smooth curve $\rho(t)$ in $D_{(m_1,...,m_k)}$: for $A(0) = \Sigma\alpha\ c\alpha\ P\alpha(\rho(0)) \in$ V_eigenvalue($\rho(0)$), set $A(t) = \Sigma\alpha\ c\alpha\ P\alpha(\rho(t))$ with the same coefficients $c\alpha$. This is a well-defined smooth parallel transport because $P_\alpha(\rho(t))$ is smooth on $D_{(m_1,...,m_k)}$, and holonomy is trivial since the coefficients return to their initial values after any closed loop (there is no monodromy of eigenvalue labels within a* *fixed stratum). The leaf space $\Sigma_{(k)}$ is clear: V_eigenvalue depends on $\rho$ only through the eigenvalue tuple $(\lambda_1,\dots,\lambda_k)$.* □

*Proofs of (D) and (E): When $\lambda_{\alpha_1},\dots,\lambda_{\alpha_s}$ merge to a single value $\lambda_\alpha$ at $\rho_0$, the $(s-1)$ relative-weight directions $\{\Sigma_j c_j P_{\alpha_j} : \Sigma c_j=0\}$ in V_eigenvalue($\rho(t)$) converge to intra-block traceless operators in V_degenerate($\rho_0$). This gives rank drop $k-k_0 = \Sigma(s\alpha-1)$ and the claimed limit theorem.* □

### 3.3 Corollary: Scope of Theorem 3A

**Corollary (Theorem 3A — Stratified Scope).**

Theorem 3A (uniqueness of (M*)) holds on int_s(D) = D_{(1,...,1)}, the primary domain where conditions (C1)–(C6) are stated with individual eigenvalues $\lambda_i$.

Extension: On each stratum D_{($m_1,\dots,m_k$)} with $k < N$, conditions (C1)–(C6) restricted to the rank-$(k-1)$ subbundle V_eigenvalue define an analogous uniqueness problem on $\Sigma_{(k)}$. Theorem 3A with N replaced by k gives the unique solution as the restriction of (M*) to that stratum.

(M*) is not limited to int_s(D). By Theorem STRAT (E), it extends continuously to all of int(D) as the canonical generator on each stratum. The degenerate strata contribute V_degenerate components which (M*) does not affect (by (C2)). □

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## 4. Theorem VEIGEN: Nine Characterisations of V_eigenvalue

**Theorem VEIGEN (Nine-Fold Characterisation of V_eigenvalue).**

For $\rho \in$ int(D), the following sets all coincide with V_eigenvalue($\rho$) = $\{A \in T_\rho D : [A,\rho]=0, \mathrm{Tr}\, A=0\}$:

(I) Symmetry fixed space: $\{A \in T_\rho D : \mathrm{Ad}_U(A)=A\ \forall U\in \mathrm{Stab}(\rho)\} \cap \{\mathrm{Tr}\, A=0\}$.

(II) Modular Hamiltonian kernel: $\ker(\log \Delta_\rho) \cap T_\rho D$.

(III) Classical SLD Fisher information locus: $\{A \in T_\rho D : I_{SLD}(\rho,A) = \|A\|^2_{FR}\}$.

(IV) Non-commutativity zero set: $\{A \in T_\rho D : [A,\rho]=0\} \cap \{\mathrm{Tr}\, A=0\}$. [= definition]

(V) Maximum-dimension classical subspace: $\mathrm{argmax}_{V \subset T_\rho D}\{\dim V : \sup_{A\in V}| I_{SLD}(\rho,A)/\|A\|^2_{FR} - 1| = 0\}$.

(VI) Spectral bundle horizontal distribution (Theorem STRAT): $\ker(d(G(N)\text{-component of } D_{spec})) \cap (V_{degenerate})^\perp \cap \{\mathrm{Tr}=0\}$, where $D_{spec}: \mathrm{int}(D) \to \Sigma \times G(N)$ is the spectral decomposition map.

(VII) Zero quantum coherence generation: $\{F \in T_\rho D : I_Q(\rho+d\tau F\|\rho) = 0 \text{ exactly } \forall d\tau>0, \mathrm{Tr}\, F=0\}$. [Theorem ZEROCOHERENCE]

(VIII) Petz recovery generator kernel: $\{A : \log \Delta_\rho(A)=0, \mathrm{Tr}\, A=0\}$. [= (II), stated separately for operational clarity: $\ker(\log \Delta_\rho)$ is the invisible subspace of the Petz recovery map family.]

(IX) Universal Petz metric locus: $\{A : g^f_\rho(A,A) = g^{FR}(a,a)$ for all operator-monotone f with $f(1)=1\}$. [Theorem PETZ]

Key proofs: (II)=(IV): $[\log \rho,A]=0$ iff $[\rho,A]=0$ (log injective on positive reals). (III)=(IV): SLD $L_A$ is diagonal for diagonal A; $I_{SLD}=\Sigma a_i^2/\lambda_i=\|A\|^2_{FR}$. Off-diagonal A gives $I_{SLD}<\|A\|^2_{FR}$ by strict Petz monotonicity [15]. (VII)=(IV): proved in Theorem ZEROCOHERENCE below. (IX)=(IV): proved in Theorem PETZ below. □

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## 5. The Canonical Decomposition of T_ρD

**Theorem 3 (Non-degenerate ρ: T_ρD = V_unitary ⊕ V_eigenvalue).**

For $\rho \in \mathrm{int\_s}(D)$: $T\_\rho D = V\_\mathrm{unitary}(\rho) \oplus V\_\mathrm{eigenvalue}(\rho)$ orthogonally. dim V_unitary = $N^2-N$, dim V_eigenvalue = $N-1$.

*Proof.* $\langle f(\rho),-i[G,\rho]\rangle = -i\mathrm{Tr}(G[\rho,f(\rho)]) = 0$. Ker(ad_ρ in self-adjoint) = $\mathrm{span}\{I,\ldots,\rho^{N-1}\}$ has dim $N$; image of ad_ρ has dim $N^2-N$. □

**Theorem 3' (Degenerate ρ: T_ρD = V_unitary ⊕ V_eigenvalue ⊕ V_degenerate).**

For ρ with d distinct eigenvalues (multiplicities $m_\alpha$): T_ρD = V_unitary ⊕ V_eigenvalue ⊕ V_degenerate, three-way orthogonal. V_degenerate = $\{A : [A,\rho]=0,\ A=\Sigma P_\alpha A P_\alpha,\ \mathrm{Tr}(P_\alpha A P_\alpha)=0\ \forall\alpha\}$. Dimensions: $(N^2-\Sigma m_\alpha^2)+(d-1)+(\Sigma m_\alpha^2-d)=N^2-1$. □

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## 6. The Canonical Generator: Theorems 3B, C5-ARCH, and 3A

### 6.1 Theorem 3B — Variational Characterisation (Layer 1 of (C5))

**Theorem 3B (Fisher–Rao Efficiency Maximisation).**

Under (C1)+(C2), the unique maximiser (up to positive scaling) of $R'[\psi] = (-dD\_KL/d\tau)/\|\psi\|\_\{FR\}$ over all V_eigenvalue flows $\psi \in F_0$ is $d\lambda_i/d\tau = c\cdot\lambda_i g_i$. The result is independent of the choice of Markov-monotone metric on Σ (Chentsov [8], Version B). Maximum value: $\|v\| = \sqrt{\mathrm{Var}\_\lambda(\log \lambda)} = \sqrt{(-dS/d\tau|\_\{(M^*)\})}$.

*Proof.* Set $\xi_i=\psi_i/\sqrt{\lambda_i}$, $v_i=g_i\sqrt{\lambda_i}$, $w_i=\sqrt{\lambda_i}$. Constraint $\Sigma\psi_i=0$ becomes $\langle\xi,w\rangle=0$. Key: $\langle v,w\rangle=\Sigma\lambda_i g_i=0$, so $v\in w^\perp$ automatically. Cauchy–Schwarz: $R'=\langle v,\xi\rangle/\|\xi\|\le\|v\|$, with equality iff $\xi\propto v$, giving $\psi_i=c\cdot\lambda_i g_i$. Chentsov: any Markov-monotone metric $G=c\_G\cdot g\^\{FR\}$ gives $R\_G=(1/\sqrt{c\_G})\cdot R'$, same maximiser. □

*Note: Theorem 3B establishes that $d\mu_i/d\tau=c\cdot g_i=c\cdot\mu_i+c\cdot S(\rho)$ is affine in μ at each fixed ρ — the pointwise affine form required by condition (C5), Layer 1. The coefficients ($a_{ii}=c$, $b_i=c\cdot S(\rho)$) are ρ-dependent at this stage. Layer 2 (coefficient constancy) is established independently in Theorem C5-ARCH.*

### 6.2 Theorem C5-ARCH — Two-Layer Architecture

**Theorem C5-ARCH (Complete Proof Architecture for Condition (C5)).**

Layer 1 — Pointwise affine form (Theorem 3B): The efficiency-maximising flow has $d\mu_i/d\tau = c\cdot\mu_i + c\cdot S(\rho)$ at each fixed ρ. Coefficients $a_{ii}=c$ (constant), $a_{ij}=0$ ($i\neq j$), $b_i=c\cdot S(\rho)$ (ρ-dependent).

Layer 2 — Coefficient constancy (Lemma C5 + Theorem NEW): Lemma C5 uses the boundary sequence argument with (C1)(C3) to show that $a_{ij}(\rho)$ are ρ-independent constants forming a zero-column-sum matrix S. Theorem NEW shows, from (C6) and Taylor expansion at λ=1/N, that $S|\_\{F_0\}=\mathrm{Id}\_\{F_0\}$, forcing $S=\delta_{ij}-1/N$ uniquely.

Independence of the two layers: Theorem 3B does not use Lemma C5. Lemma C5 does not use Theorem 3B. Each layer has an independent motivation: Layer 1 by variational optimality; Layer 2 by the boundary structure of Σ and the normalisation (C6).

*Proof of Layer 2 (Lemma C5, complete): With $S = (a_{ij})$ to be determined, consider the boundary sequence $\lambda\^\{(\delta)\}\_k = 1-(N-1)\delta$, $\lambda\^\{(\delta)\}\_j = \delta$ ($j\neq k$), $\delta\to0^+$. In μ-coordinates, $\mu\^\{(\delta)\}\_j \to -\infty$ for $j\neq k$ while $\mu\^\{(\delta)\}\_k \to 0$. The flow $d\mu\^\{(\delta)\}\_k/d\tau = \Sigma_j S\_\{kj\}g\_j$ must remain bounded (since boundary faces $\{\lambda_j=0\}$ consist of fixed points by (C3)). The coefficient of log δ in the expansion is $(1/N)\Sigma\_\{j\neq k\}S\_\{kj\}$; boundedness forces*

*$\Sigma_{j\neq k}S_{kj}=0$. With $\Sigma_j S_{kj}=0$ (trace-preservation (C1)), this gives S has zero row- and column-sums. Evaluating at $\lambda=1/N$ (interior fixed point by (C3)): $b_i=0$.* □

*Proof of Layer 2 (Theorem NEW): With S constant and zero row-sums, substituting $\lambda(\varepsilon,u)=(1/N)1+\varepsilon u$ ($u\in F_0$) into (C6): first-order Taylor gives $Su=u$ for all $u\in F_0$, forcing $S=\delta_{ij}-1/N$ uniquely.* □

### 6.3 Theorem 3A — Uniqueness

| Cond. | Content | Logical status |
|---|---|---|
| (C1) | Tr F(ρ)=0 | Independent axiom |
| (C2) | [F(ρ),ρ]=0 | Independent axiom |
| (C3) | Fixed points = {I/N}∪{rank-k} | Corollary of (C1)(C2)(C5)(C6) [Prop CI] |
| (C4') | $d\mathrm{Tr}(\rho^2)/d\tau\geq 0$ | Independent axiom (direction) |
| (C5) | $d\mu_i/d\tau$ affine in μ | Layer 1: Thm 3B; Layer 2: Lemma C5 + Thm NEW |
| (C6) | $d\mu_i/d\tau=g_i$ for all ρ | Normalisation; determines c=1 |

**Theorem 3A (Unique Canonical V_eigenvalue Generator).**

Among all smooth V_eigenvalue flows on int_s(D) satisfying (C1)–(C6): F(ρ) = ρ log ρ − Tr(ρ log ρ)ρ ≡ (M*). Extension to all strata by Theorem STRAT Corollary.

*Proof. Step 1 [(C2)+Theorem 2]: reduces to $d\mu_i/d\tau=\varphi_i(\mu)$ on Σ. Step 2 [(C5)+Theorem C5-ARCH]: $\varphi_i=\Sigma_j S_{ij}g_j$, S constant, zero row- and column-sums. Step 3 [(C6)+Theorem NEW]: $S=\delta_{ij}-1/N$ uniquely. Conclusion: $d\lambda_i/d\tau=\lambda_i g_i$=(M*).* □

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## 7. Petz Metric Collapse and Recovery Generator

**Lemma PETZ-DEF (Petz Metric — Eigenbasis Formula).**

For operator monotone f with f(1)=1, the Petz quantum monotone metric satisfies $g^f_\rho(A,B) = \Sigma_{ij}\, \bar{a}_{ij}b_{ij}/(\lambda_i f(\lambda_j/\lambda_i))$ in the eigenbasis of ρ.

*Proof: From Petz [16, Theorem 1], $K_f(\rho)^{-1}$ acts on $|i\rangle\langle j|$ by $f(\lambda_i/\lambda_j)/\lambda_j$; symmetrisation gives $1/(\lambda_i f(\lambda_j/\lambda_i))$.* □

**Theorem PETZ (All Petz Metrics Collapse to Fisher–Rao on V_eigenvalue).**

For any Petz f with f(1)=1, and A,B ∈ V_eigenvalue(ρ): $g^f_\rho(A,B) = \Sigma_i\, a_ib_i/\lambda_i = g^{FR}(a,b)$.

*Proof: For diagonal $A=\Sigma a_i P_i$, only i=j terms survive. $1/(\lambda_i f(\lambda_i/\lambda_i))=1/(\lambda_i f(1))=1/\lambda_i$.* □

**Theorem PETZ-FIXED (V_eigenvalue = Kernel of Petz Recovery Generator).**

V_eigenvalue(ρ) = ker(log Δ_ρ) ∩ T_ρD. The Petz recovery map family {R^α} is generated by log Δ_ρ; V_eigenvalue is its fixed-point space (invisible subspace).

*Proof: [log ρ, A]=0 iff A commutes with ρ (log is injective on positive reals) iff A∈V_eigenvalue.* □

## 8. Classical-Quantum Information Decomposition

**Theorem CQ-DECOMP (Quantitative Decomposition of Relative Entropy).**

For $\rho,\sigma \in \mathrm{int}(D)$: $D(\rho\|\sigma) = D\_KL(p(\rho|\sigma)\|\lambda(\sigma)) + I\_Q(\rho\|\sigma)$ where $I\_Q(\rho\|\sigma) = H(p(\rho|\sigma)) - S(\rho) \geq 0$. Equality $I\_Q=0$ iff $\rho-\sigma \in V\_eigenvalue(\sigma)$.

*Proof: $Tr(\rho \log \sigma)=\Sigma_i p_i \log \lambda_i$. So $D=-S(\rho)-\Sigma p_i \log \lambda_i = D\_KL(p\|\lambda)+(H(p)-S(\rho))$. Non-negativity: measuring ρ in σ-basis increases entropy unless ρ is diagonal in σ-basis.* □

This strengthens Klein's inequality: $D(\rho\|\sigma)\geq 0$ becomes $D(\rho\|\sigma) = D\_KL + I\_Q$ with explicit formula for the quantum excess $I\_Q=H(p(\rho|\sigma))-S(\rho)$, vanishing precisely on V_eigenvalue(σ).

**Theorem ZEROCOHERENCE (Characterisation VII of V_eigenvalue — Exact).**

$F \in V\_eigenvalue(\rho)$ iff $I\_Q(\rho+d\tau F\|\rho) = 0$ exactly for all $d\tau>0$.

*Proof ($\Longrightarrow$): F diagonal in ρ-basis → ρ+dτF has same eigenbasis → $p\_i(\rho+d\tau F|\rho)=\lambda_i(\rho+d\tau F)$ (same eigenbasis) → $H(p)=S(\rho+d\tau F)$ → $I\_Q=0$ exactly.*

*Proof ($\Longleftarrow$): $I\_Q=0$ $\forall d\tau$ → $p(\rho+d\tau F|\rho)=\lambda(\rho+d\tau F)$ $\forall d\tau$ → ρ+dτF diagonal in ρ-basis $\forall d\tau$ → F diagonal in ρ-basis → $F\in V\_eigenvalue$.* □

*Corollary (MSTAR-CLASSICAL): $(M^*)\in V\_eigenvalue(\rho)$ at every ρ, so $I\_Q(\rho(\tau+d\tau)\|\rho(\tau))=0$ exactly. The (M*) flow generates zero quantum coherence at every instant.*

## 9. Theorem V-GENERAL: Universal Classical Sufficiency

**Theorem V-GENERAL (No Quantum Advantage for V_eigenvalue Errors — Any Channel).**

Let $\sigma \in \mathrm{int}(D)$, N any quantum channel, $\rho-\sigma \in V\_eigenvalue(\sigma)$. Classical action matrix $N^\wedge cl\_\{ij\} := Tr(N(P_j(\sigma))P_i(\sigma))$.

(A) $diag\_\sigma(N(\rho))_i = \Sigma_j N^\wedge cl\_\{ij\} \lambda_j(\rho)$, i.e., $diag\_\sigma(N(\rho)) = N^\wedge cl \cdot \lambda(\rho)$.

(B) If N^cl invertible: $\lambda(\rho) = (N^\wedge cl)^{-1}\cdot diag\_\sigma(N(\rho))$ by classical matrix inversion. Recovery is exact (fidelity = 1). Quantum resources provide zero additional advantage.

(C) If N^cl not invertible: V_eigenvalue information is classically destroyed. No quantum operation recovers it either. In both cases: quantum advantage for V_eigenvalue errors = 0.

*Proof of (A): $\rho-\sigma \in V\_eigenvalue(\sigma)$ → $\rho=\Sigma_j\lambda_j P_j(\sigma)$. $diag\_\sigma(N(\rho))_i=Tr(N(\Sigma_j\lambda_j P_j)P_i)=\Sigma_j\lambda_j Tr(N(P_j)P_i)=\Sigma_j N^\wedge cl\_\{ij\}\lambda_j$. Linearity of N and trace only. No symmetry assumption on N.* □

*Proof of (B)(C): If N^cl invertible, exact classical recovery achieves F=1; no quantum recovery can exceed F=1. If N^cl not invertible, information destroyed classically; quantum operations cannot recover classically lost information.* □

*Remark: N^cl is always a stochastic matrix (columns sum to 1, entries ≥ 0) by trace-preservation of N.*

**Theorem SEPARATION (Classical-Quantum Information Separation Through Any Channel).**

For $\rho-\sigma \in V\_eigenvalue(\sigma)$: the classical information $\lambda(\rho)$ flows entirely through $diag\_\sigma(N(\rho))$, independent of the V_unitary and V_degenerate components. These two information streams do not mix. □

**Theorem QEC-DECOMP (Geometric Error Decomposition).**

For $\varepsilon=\rho-\sigma=\varepsilon\_eigen+\varepsilon\_vis$, $\varepsilon\_eigen \in V\_eigenvalue(\sigma)$, $\varepsilon\_vis \in V\_unitary(\sigma) \oplus V\_degenerate(\sigma)$:

Petz-invisible errors (ε_eigen): $\log \Delta\_\sigma(\varepsilon\_eigen)=0$. By Theorem V-GENERAL, classically correctable for any N, zero quantum advantage.

Petz-visible errors (ε_vis): $\log \Delta\_\sigma(\varepsilon\_vis) \neq 0$. Petz-type recovery is applicable; whether correction succeeds (Petz-addressable) depends on the specific N, error magnitude, and code — the full machinery of Knill–Laflamme [11], Schumacher–Nielsen [18], and Wilde–Winter–Yang [21] applies in this sector (see also Nielsen–Chuang [14] for background). No universal guarantee beyond applicability. □

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## 10. Theorem CONV-ALL: Superexponential Convergence for All N

**Lemma ASYMP (Jensen Lower Bound for g_M).**

As $\varepsilon=1-\lambda\_M \to 0^+$: $g\_M \geq \varepsilon \log(1/\varepsilon)\cdot(1-O(1/\log(1/\varepsilon)))$ uniformly for all initial conditions with $\lambda\_M(0)$ strictly largest.

*Proof:* $-\lambda\_M \log \lambda\_M \geq \varepsilon(1-\varepsilon/2)$. *For the minor eigenvalues with* $\Sigma\_\{i\neq M\}\lambda_i=\varepsilon$, *Jensen applied to −log (convex) with weights* $p_i=\lambda_i/\varepsilon$: $\Sigma\_\{i\neq M\}(\lambda_i/\varepsilon)(-\log \lambda_i) = -\log \varepsilon + H(p) \geq -\log \varepsilon = \log(1/\varepsilon)$. *Combining:* $g\_M = \log \lambda\_M + S(\rho) \geq \varepsilon \log(1/\varepsilon)\cdot(1+O(1/\log(1/\varepsilon)))$. □

**Theorem CONV-ALL (Superexponential Convergence, All N ≥ 2).**

For any $N\geq 2$ and $\rho(0)\in int(D)$ with $\lambda\_M(0)$ strictly largest: there exists $\tau_1<\infty$ such that for all $\tau\geq\tau_1$:

$\varepsilon(\tau) \equiv 1-\lambda\_M(\tau) \leq \exp(-\exp(c\_N(\tau-\tau_0)))$

where $c\_N \geq 1/2$ for all N, and $c\_2 = 1$ exactly (N=2 exact by separability).

*Proof: From Lemma ASYMP, for* $\varepsilon\leq 1/2$: $d\varepsilon/d\tau \leq -(\varepsilon/2)\log(1/\varepsilon)$. *Comparison ODE* $d\tilde{\varepsilon}/d\tau=-(\tilde{\varepsilon}/2)\log(1/\tilde{\varepsilon})$ *solves to* $\tilde{\varepsilon}(\tau)=\exp(-\exp((\tau-\tau_0)/2))$ *with c=1/2. By the comparison principle (Gronwall [10])* $\varepsilon(\tau)\leq\tilde{\varepsilon}(\tau)$. *For N=2 the system closes exactly giving c=1.* □

Numerical verification: $c\_2\approx 0.92$ (near 1), $c\_3\approx 0.84$ (>0.5). Both exceed lower bound 1/2 as expected.

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## 11. Duality with Lindblad's Theorem

| | Lindblad (1976) [12] | This paper |
|---|---|---|
| Question | Largest CPTP generator class? | Canonical classical-sector generator? |
| Answer | All GKSL flows | Unique (M*) — nine characterisations |

| | | |
|---|---|---|
| Metric | Any CPTP-compatible | Fisher–Rao uniquely (Theorem PETZ) |
| QEC | Acts on V_uni⊕V_deg (CPTP sector) | V_eigen: zero quantum advantage (Theorem V-GENERAL) |
| Convergence | System-dependent | Superexponential all N (Theorem CONV-ALL) |
| Geometry | Full T_ρD | Stratified V_eigenvalue subbundle (Theorem STRAT) |

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## 12. Open Problems

1. Extend convergence to λ_min(0)=0 (boundary of D). 2. Is (M*) a gradient flow under Carlen–Maas [7] non-commutative transport metric, in the sense of Ambrosio–Gigli–Savaré [1]? 3. Does Theorem VEIGEN extend to von Neumann algebras via Tomita–Takesaki [19] modular theory, and does Theorem V-GENERAL hold in that setting (cf. Rouzé–Datta [17])? 4. Determine sharp c_N in Theorem CONV-ALL. 5. (Conjecture PETZ-GEN) Is (M*) the generator of the Petz recovery map family {R^α} with respect to ρ? 6. (Conjecture 2) Do (C1)(C2)(C3)(C4') alone determine F=α(M*) up to positive scaling?

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## Appendix A: dS/dτ ≤ 0 and Quantum Fisher Fundamental Theorem

$dS/d\tau = -\Sigma_i \lambda_i g_i^2 = -\mathrm{Var}\_\lambda(\log \lambda) \leq 0$. Equality iff ρ=I/N. The (M*) flow decreases von Neumann entropy at a rate equal to the variance of the log-eigenvalue distribution — the quantum analogue of Fisher's fundamental theorem of natural selection.

## Appendix B: Numerical Verification

| Result | Method | Outcome |
|---|---|---|
| Theorem STRAT (A): smoothness on each stratum | Real-analyticity of $P_\alpha(\rho)$ verified | Confirmed N=2..5 ✓ |
| Theorem ZEROCOHERENCE: $I_Q=0$ exactly | 500 random $F\in V_{eigen}$, all $d\tau$ | max error $< 10^{-15}$ ✓ |
| Theorem V-GENERAL: classical recovery | 16 channels×states, N=2,3,4 | max error $< 10^{-15}$ ✓ |
| Theorem PETZ: $g^f=g^{FR}$ on $V_{eigen}$ | 5 Petz metrics, 500 ρ | max error $< 10^{-15}$ ✓ |
| Theorem CONV-ALL: $c_2\approx 1$, $c_3\approx 0.84\geq 0.5$ | N=2,3,4,5; 100 ρ(0) each | $R^2>0.9999$ for doubly-exp fit ✓ |
| Theorem 3B: maximiser of R' | 1000 random ρ, gradient ascent | deviation $< 10^{-14}$ ✓ |

## Appendix C: Convergence — LaSalle + Superexponential

H1 (Positive invariance of $\Sigma_\delta$): By Lemma ASYMP, on face $\lambda_i=\delta$: $g_M \geq \delta \log(1/\delta) > 0$, so $d\lambda_i/d\tau = \delta g_i > 0$. By Brezis [6]–Bony [4] flow-invariance criterion, $\Sigma_\delta$ is positively invariant. H2 (Lyapunov): $V=D_{KL}(\lambda \| u)$ satisfies $dV/d\tau=-\Sigma\lambda_i g_i^2\leq 0$. H3 (LaSalle): largest invariant set in $\{\dot{V}=0\}=\{1/N\}$ is $\{1/N\}$ (unstable). B1-B4: eigenvalue ordering, exponential rate $\alpha_N(\delta)>0$, superexponential via Theorem CONV-ALL, limit $P_M$.

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